CHARACTERIZING ABELIAN ADMISSIBLE GROUPS

J. BRUNA, J. CUFÍ, H. FÜHR, M. MIRÓ

Abstract. By definition, admissible matrix groups are those that give rise to a wavelet-type inversion formula. This paper investigates necessary and sufficient admissibility conditions for abelian matrix groups. We start out by deriving a block diagonalization result for commuting real valued matrices. We then reduce the question of deciding admissibility to the subclass of connected and simply connected groups, and derive a general admissibility criterion for exponential solvable matrix groups. For abelian matrix groups with real spectra, this yields an easily checked necessary and sufficient characterization of admissibility. As an application, we sketch a procedure how to check admissibility of a matrix group generated by finitely many commuting matrices with positive spectra.

We also present examples showing that the simple answers that are available for the real spectrum case fail in the general case.

An interesting byproduct of our considerations is a method that allows for an abelian Lie-subalgebra $\mathfrak{h} \subset gl(n,\mathbb{R})$ to check whether $H = \exp(\mathfrak{h})$ is closed.

1. Introduction

Let us quickly recall the group-theoretic formalism for the construction of continuous wavelet transforms in higher dimensions. For a more complete introduction, we refer to [9, 15]. The starting point is a subgroup $H < \mathrm{GL}(n,\mathbb{R})$, called the *dilation group*. Its action on $\mathbb{R}^n$ gives rise to the semidirect product $G = \mathbb{R}^n \rtimes H$, which is just the group of affine mappings generated by H and all translations. We write elements of this group as (x,h), with $x \in \mathbb{R}^n$, $h \in H$. This group acts unitarily on the Hilbert space $\mathrm{L}^2(\mathbb{R}^n)$ via the *quasiregular representation*

$$\pi(x,h)f(y) = |\det(h)|^{-1/2} f(h^{-1}(y-x)) \ .$$

The associated continuous wavelet transform of $f \in \mathrm{L}^2(\mathbb{R})$ is obtained by picking a suitable $\psi \in \mathrm{L}^2(\mathbb{R})$ and letting

$$V_\psi f(x,h) = \langle f, \pi(x,h)\psi\rangle = \int_{\mathbb{R}^n} f(y)|\det(h)|^{-1/2}\overline{\psi\left(h^{-1}(y-x)\right)}dy \ \text{ for } (x,h) \in G \ .$$

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The wavelet ψ is called *admissible* if $V_\psi : \mathrm{L}^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathrm{G})$ is isometric. In this case, we have the *wavelet inversion formula*

$$f(y) = \int_H \int_{\mathbb{R}^n} V_\psi f(x,h) |\det(h)|^{-1/2} \psi\left(h^{-1}(y-x)\right) dx \frac{dh}{|\det(h)|} ,$$

to be read in the weak sense (rather than pointwise), where dh is the left Haar measure on H. The matrix group H is called *admissible* if there exists an admissible vector in $\mathrm{L}^2(\mathbb{R}^n)$.

Necessary and sufficient criteria for admissibility have been studied with increasing generality since the early nineties; see eg. [18, 4, 3, 7, 8, 9, 15]. A complete characterization of admissibility in terms of the dual action appeared quite recently [11]. The existing literature already suggests a large variety of admissible matrix groups. This paper undertakes a more systematic study of admissibility for the subclass of abelian matrix groups. These groups were previously studied in [8, 14], with the former source focussing on the dilation groups H for which π is a finite sum of irreducibles, and the latter on one-parameter groups. Already in the restricted setting of [8], the class of admissible matrix groups quickly becomes too large to manage: It turns out that the conjugacy classes of admissible abelian matrix groups for which the quasiregular representation is irreducible are in natural correspondence to the isomorphism classes of commutative algebras with unity of the same dimension up to isomorphism. In particular, a classification modulo conjugacy of this subclass is out of sight.

Thus it seems that the best one can hope for are methods that allow, for a concrete matrix group described by finite data (e.g., by –possibly infinitesimal– generators), to decide admissibility in an algorithmic way. Our aim is to derive sufficient conditions that are easy to check and widely applicable. In particular, we want to address finitely generated abelian dilation groups: Given any set of pairwise commuting matrices $A_1, \ldots, A_k$, when does there exist a $\psi \in \mathrm{L}^2(\mathbb{R}^n)$ guaranteeing the wavelet inversion formula

$$f(y) = \sum_{\ell \in \mathbb{Z}^k} \int_{\mathbb{R}^n} V_\psi f(x, A^\ell) |\det(A^\ell)|^{-1/2} \psi\left(A^{-\ell}(y-x)\right) dx \ ?$$

Here we used the multi-index notation $A^\ell = A_1^{\ell_1} \cdot \ldots \cdot A_k^{\ell_k}$. The interest in discrete groups is amplified by the following observation: Suppose that, for a suitable lattice $\Gamma \subset \mathbb{R}^n$ the system $(\pi(x, A^\ell)\psi)_{x \in \Gamma, \ell \in \mathbb{Z}^k}$ is a wavelet frame, i.e., one has for all $f \in \mathrm{L}^2(\mathbb{R}^n)$ that

$$A\|f\|_2^2 \le \sum_{x,\ell} \left|\langle f, \pi(x, A^\ell)\psi\rangle\right|^2 \le B\|f\|_2^2 .$$

Then the results of [1] imply that $H = \langle A_1, \ldots, A_k \rangle$ is admissible. Thus the methods developed here will allow to derive necessary criteria for dilation groups generating a wavelet frame.

Let us now give a short outline of the paper and the main results. Section 2 contains an overview of admissible matrix groups and the criteria characterizing them. Section 3 is concerned with the structure of commuting matrices. Theorem 4 is the main structure result in this section, describing a form block diagonalization of commuting matrices. It

is probably well-known; we include a full proof to stress that the bases corresponding to the block diagonalization can be computed via the Gauss algorithm, if the spectra of the commuting matrices are known. As an application of the structure result, we consider the question whether the exponential image of an abelian matrix Lie algebra is closed. It turns out that once the spectra of a set of generators are known, this question can be decided by repeated applications of the Gauss algorithm (Theorem 15). This theorem, and some of the auxiliary results leading up to it, can be considered to be of independent interest, but they are also crucial for the subsequent results, in particular for the proof and construction of counterexamples in the final section of this paper. In Section 4, we return to the discussion of admissible groups. Here, the chief result is Proposition 17, reducing the problem of deciding admissibility for arbitrary abelian matrix groups to the subclass of simply connected, connected groups. For this setting, the admissibility criteria from Section 2 are investigated more closely in Section 5. It turns out that for connected abelian matrix groups with real spectrum, there exist simple computational criteria allowing to decide admissibility (Corollary 31, in conjunction with Proposition 21). Section 6 applies these results to describe a procedure for checking admissibility of a matrix group generated by finitely many commuting matrices with positive spectra. Finally, Section 7 gives examples showing that the criteria established for the real spectrum case can fail in the general case.

2. Admissible matrix groups

Throughout the paper, $GL(n,\mathbb{R})$ denotes the group of invertible real-valued $n \times n$ matrices, $gl(n,\mathbb{R})$ the vector space of all matrices. A *matrix group* is a subgroup of $GL(n,\mathbb{R})$. For a Lie-subgroup of H of $GL(n,\mathbb{R})$, the Lie-algebra is the subspace $\mathfrak{h} \subset gl(n,\mathbb{R})$ of tangent vectors of curves in H through identity, endowed with the usual matrix commutator.

The dual group $\widehat{\mathbb{R}^n}$ is the character group of $\mathbb{R}^n$ suitably identified with the space of row vectors. In the following, the Fourier transform of $f \in \mathrm{L}^1(\mathbb{R}^n)$ is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle \xi, x \rangle} dx \ .$$

Admissibility of a matrix group H is closely related to its *dual action*. It can be understood as a right action on the Fourier side. Given a dilation group $H < \mathrm{GL}(n,\mathbb{R})$ and $\psi \in \mathrm{L}^2(\mathbb{R}^n)$, admissibility of ψ is equivalent to the *Calderon condition* [9, 15]

$$\int_H |\widehat{\psi}(\omega h)|^2 dh = 1 \ , \text{a.e. } \omega \in \widehat{\mathbb{R}^n} \ . \tag{1}$$

A particularly well-studied case concerns the existence of *open dual orbits*, which is closely related to so-called *discrete-series representations*, see [3, 7]. In this case, $\widehat{\mathbb{R}^n}$ is the union of finitely many orbits (up to a set of measure zero), and admissible vectors exist iff all open orbits have associated compact stabilizers [7].

We will now present an overview of known properties of admissible matrix groups. We first recall an observation from [8]:

Lemma 1. *Let $H < \mathrm{GL}(n, \mathbb{R})$ be a Lie subgroup. If it is admissible, it is closed.*

In the general setting, the following characterization of admissible matrix groups can be derived. For the formulation, given $\omega \in \widehat{\mathbb{R}^n}$ let $\mathrm{stab}_H(\omega)$ denote the stabilizer of ω under the dual action of H in $\widehat{\mathbb{R}^n}$. Given a Borel H-invariant subset $U \subset \widehat{\mathbb{R}^n}$, a subset $C \subset U$ meeting each H-orbit in precisely one point is called a *fundamental domain.* We call a subset $C \subset \widehat{\mathbb{R}^n}$ a *measurable fundamental domain* if C is a Borel subset and a fundamental domain for an H-invariant Borel subset of full measure. The following theorem is [11, Theorem 6]; see [17] for further equivalent formulations of the criterion.

Theorem 2. *Let $H < \mathrm{GL}(n, \mathbb{R})$ be given. Then the following are equivalent:*

(a) *H is admissible.*
(b) *The following three conditions are fulfilled:*
  (i) *There exists a measurable fundamental domain of $\widehat{\mathbb{R}^n}$.*
  (ii) *For almost all $\omega \in \widehat{\mathbb{R}^n}$, $\mathrm{stab}_H(\omega)$ is compact.*
  (iii) *There exists $h \in H$ such that $|\det(h)| \neq \Delta_H(h)$.*

Here Δ_H denotes the modular function of H. For abelian groups H, note that $\Delta_H(h) \equiv 1$.

Regarding the compact stabilizer condition, let us note the following facts:

Lemma 3. *Let H be a closed matrix group. Then:*

(a) *The set $\Omega_c = \{\omega \in \widehat{\mathbb{R}^n} : \mathrm{stab}_H(\omega)$ is compact $\}$ is a Borel subset of $\widehat{\mathbb{R}^n}$.*
(b) *Suppose that H is discrete. Then Ω_c is conull.*
(c) *Suppose that H is simply connected, connected and abelian. Then H acts freely on Ω_c.*

Proof. For part (a), we refer to [10, 5.6]. For part (b), note that $h \in \mathrm{stab}_H(\omega)$ if and only if ω belongs to the eigenspace of h associated to the eigenvalue 1. For $h \neq \mathbf{1}$, this eigenspace is proper, and the union over all associated eigenspaces is a set of measure zero. For ω outside this set, the stabilizer is trivial. For part (c), note that $H \cong \mathbb{R}^k$, and thus H has no compact subgroups. □

3. Structure of commuting matrices

It is well-known that, given a set of commuting matrices over the complex numbers, there exists a basis with respect to which all matrices have upper triangular form. In this section, we will derive a similar result over the reals. Here, upper tridiagonalization generally cannot be fully achieved. However, an approach that is similar to the derivation of the real Jordan form allows to formulate a useful replacement.

The formulation of the result requires additional terminology. We write $\mathbb{K}$ for an element of $\{\mathbb{R}, \mathbb{C}\}$. We will often use the natural embedding $gl(n, \mathbb{R}) \subset gl(n, \mathbb{C})$. On the other hand, it will be convenient to identify complex-valued matrices with real-valued ones

with double dimensions: Given a matrix $A \in gl(n,\mathbb{C}), A = (a_{i,j})_{i,j}$, we denote by $i_{\mathbb{C}}(A)$ the $2n \times 2n$-matrix obtained by replacing each complex entry $a_{i,j}$ by the real matrix $\begin{pmatrix} \mathrm{Re}(a_{i,j}) & -\mathrm{Im}(a_{i,j}) \\ \mathrm{Im}(a_{i,j}) & \mathrm{Re}(a_{i,j}) \end{pmatrix}$. Thus $i_{\mathbb{C}} : gl(n,\mathbb{C}) \to gl(2n,\mathbb{R})$ is an $\mathbb{R}$-algebra monomorphism, and we identify $gl(n,\mathbb{C})$ with its image. Furthermore, we let $\mathcal{N}(n,\mathbb{K})$ denote the subspace of proper upper triangular matrices over $\mathbb{K}$.

Given a matrix $A \in gl(n,\mathbb{C})$, we let $\mathrm{spec}(A)$ denote its set of eigenvalues.

The following structure result will be useful for the study of properties of abelian matrix groups.

Theorem 4. *Let $A_1, \ldots, A_k \in gl(n,\mathbb{R})$ be commuting matrices. Then there exists $B \in GL(n,\mathbb{R})$, $d_r \in \mathbb{N}$ and $\mathbb{K}_r \in \{\mathbb{R},\mathbb{C}\}$ (for $r = 1,\ldots,\ell$) such that*

$$\sum_{r=1}^{\ell} d_r \cdot \dim_{\mathbb{R}} \mathbb{K}_r = n$$

and, for $j = 1,\ldots,k$,

$$BA_jB^{-1} = \begin{pmatrix} A_{j,1} & 0 & \ldots & 0 \\ 0 & A_{j,2} & 0\ldots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \ldots & \ldots & A_{j,\ell} \end{pmatrix} \tag{2}$$

with blocks

$$A_{j,r} \in \mathbb{K}_r \cdot \mathbf{1}_{d_r} + \mathcal{N}(d_r,\mathbb{K}_r) \ .$$

If the spectra of $A_1,\ldots,A_k$ are known, B is explicitly computable by repeated applications of Gauss elimination.
One has $\mathbb{K}_1 = \ldots = \mathbb{K}_\ell = \mathbb{R}$ iff $\mathrm{spec}(A_r) \subset \mathbb{R}$ for all $1 \leq r \leq \ell$.

The existence of B can be concluded from the structure results for maximal abelian matrix algebras, as developed in [19]. However, the explicit calculation of B is not addressed in that source; to begin with, we would have to compute a maximal abelian matrix algebra containing the A_j. For this reason, we give a full proof. The proof strategy consists in first computing a decomposition of $\mathbb{C}^n$ into a sum of invariant subspaces V_i with the property, that the restriction of each A_j has a single complex eigenvalue. We then compute bases of the V_i that triangularize these restrictions. Real-valued bases are constructed by combining bases of suitable pairs of V_i. The union of these bases then gives the columns of B.

Definition 5. Let $A \in gl(n,\mathbb{K})$, $\lambda \in \mathbb{K}$ and a subspace $V \subset \mathbb{K}^n$ be given. We write

$$N(A,\lambda,V) = \{v \in V : (A - \lambda\mathbf{1})^n(v) = 0\} \ .$$

We let $N(A,\lambda) = N(A,\lambda,\mathbb{K}^n)$. Given tuples $\mathbf{A} = (A_1,\ldots,A_k)$ of matrices and $\lambda \in \mathbb{K}^k$, we define

$$N(\mathbf{A},\lambda) = \bigcap_{j=1}^{k} N(A_j,\lambda_j) \ .$$

□

Lemma 6. *Let $A \in gl(n,\mathbb{K})$. Suppose that V is an A-invariant subspace and let $V = \bigoplus_i V_i$ be a direct sum decomposition into A-invariant subspaces. Then*

$$N(A,\lambda,V) = \bigoplus_i N(A,\lambda,V_i) \ .$$

Proof. The inclusion $\supset$ is clear. For the other direction, suppose that $v \in N(A,\lambda,V)$, and $v = \sum_i v_i$. Then

$$0 = (A - \lambda \mathbf{1})^n(v) = \sum_i (A - \lambda \mathbf{1})^n (v_i)$$

with $(A - \lambda \mathbf{1})^n(v_i) \in V_i$, since V_i is invariant. Now directness of the sum implies that $v_i \in N(A,\lambda,V_i)$. □

Lemma 7. *Let $A_1,\ldots,A_k \in gl(n,\mathbb{C})$ be pairwise commuting with known spectra.*

(a) *For any given matrix B commuting with $A_1,\ldots,A_k$ and $\lambda \in \mathbb{C}^k$, the space $N(\mathbf{A},\lambda)$ is B-invariant.*

(b) *Let $\Lambda_{\mathbf{A}} = \prod_{j=1}^k \mathrm{spec}(A_j)$. Then*

$$\mathbb{C}^n = \bigoplus_{\lambda \in \Lambda_{\mathbf{A}}} N(\mathbf{A},\lambda) \ ,$$

where some of the spaces on the right hand side may be trivial.

(c) *For each $\lambda \in \Lambda_{\mathbf{A}}$, a basis of $N(\mathbf{A},\lambda)$ is computable via repeated Gauss elimination steps.*

(d) *If $\lambda \in \mathbb{R}^k \cap \Lambda_{\mathbf{A}}$ and $A_1,\ldots,A_k \in gl(n,\mathbb{R})$, then the basis from (c) can be computed with real entries.*

(e) *If $A_1,\ldots,A_k \in gl(n,\mathbb{R})$, the mapping $\mathbb{C}^n \ni v \mapsto \overline{v}$ (componentwise complex conjugation) induces a bijection $N(\mathbf{A},\lambda) \to N(\mathbf{A},\overline{\lambda})$.*

Proof. It is easily seen that $N(A,\lambda)$ is B-invariant if $[A,B]=0$. This implies (a), since the intersection of invariant spaces is invariant.

The proof of part (b) is a straightforward induction argument. The base case $k=1$ is the first step in the Jordan decomposition of A_1. In the induction step, let $A_1,\ldots,A_{k+1}$ be given. Let $\mathbf{A} = (A_1,\ldots,A_{k+1})$ and $\mathbf{A}' = (A_1,\ldots,A_k)$. The induction hypothesis yields

$$\mathbf{C}^n = \bigoplus_{\lambda' \in \Lambda_{\mathbf{A}'}} N(\mathbf{A}',\lambda') \ .$$

Note that each subspace on the right-hand side is A_{k+1}-invariant. Lemma 6 implies for $\lambda \in \mathrm{spec}(A_{k+1})$ that

$$\begin{aligned} N(A_{k+1},\lambda) &= \bigoplus_{\lambda' \in \Lambda_{\mathbf{A}'}} N(\mathbf{A}',\lambda') \cap N(A_{k+1},\lambda) \\ &= \bigoplus_{\lambda \in \Lambda_{\mathbf{A}}, \lambda_{k+1} = \lambda} N(\mathbf{A},\lambda). \end{aligned}$$

Now taking the sum over $\lambda \in \mathrm{spec}(A_{k+1})$ yields the desired decomposition.

The bases in (c) are best computed simultaneously, and again by induction over k. We start out by computing bases of the eigenspaces $N(A_1,\lambda)$, for $\lambda \in \mathrm{spec}(A_1)$. Recall that the Gauss algorithm can be employed to compute a basis of a kernel of a given matrix. This settles the case $k=1$. In the induction step, note that the computations so far provide bases of all $N(\mathbf{A}',\lambda')$, which are A_{k+1}-invariant. Therefore, A_{k+1} already has block diagonal form with respect to the basis (and the blocks are computable), and it remains to compute bases for the generalized eigenspaces of the blocks.

Since the kernel of a real-valued matrix (viewed as an operator on $\mathbb{C}^n$) has a real-valued basis, we find that the proof of (c) yields part (d) as a byproduct.

Finally, part (e) follows from the simple observation that

$$(A_i - \overline{\lambda}_i \mathbf{1})^n(\overline{v}) = 0 \Leftrightarrow \overline{(A_i - \lambda_i \mathbf{1})^n(v)} = 0 \ .$$

□

The following lemma recalls the well-known fact that pairwise commuting matrices can be jointly upper triangularized over the complex numbers. See, for instance, [20, Theorem 3.7.3] for the more general case of solvable Lie algebras. The proof therefore emphasizes the explicit computability of the matrix B using only standard linear algebra tools.

Lemma 8. *Let $A_1,\ldots,A_k \in gl(n,\mathbb{C})$ be pairwise commuting. There exists an explicitly computable $B \in GL(n,\mathbb{C})$ such that, for $j=1,\ldots,k$, the matrix BA_jB^{-1} is upper triangular. If* $\mathrm{spec}(A_j) \subset \mathbb{R}$, *for all $j=1,\ldots,k$, the matrix B can be chosen in $GL(n,\mathbb{R})$.*

Proof. One first proves by induction over k that $A_1,\ldots,A_k$ share a nontrivial eigenvector which is explicitly computable. This being trivial for $k=1$, assume that v is a eigenvector of $A_1,\ldots,A_{k-1}$ with eigenvalues $\lambda_1,\ldots,\lambda_{k-1}$, respectively, meaning that the subspace $E=\{u : A_j u = \lambda_j u,\ j=1,\ldots,k-1\}$ is non-trivial. Now, E is easily seen to be invariant by A_k, because $[A_i,A_j]=0$, whence A_k restricted to E has a non-trivial eigenvector.

We prove now the lemma by induction over n; the case $n=1$ being trivial. Let b_n denote a common eigenvector of $A_1,\ldots,A_k$ and compute vectors $v_1,\ldots,v_{n-1}$ such that $v_1,\ldots,v_{n-1},b_n$ is a basis. Writing these vectors into a matrix yields an invertible matrix B_0 such that

$$B_0 A_j B_0^{-1} = \begin{pmatrix} \tilde{A}_j & y_j \\ 0 & \lambda_j \end{pmatrix} ,$$

with $\tilde{A}_j \in gl(n-1,\mathbb{C})$ and $y_j \in \mathbb{C}^{n-1}$. The $\tilde{A}_j$ are pairwise commutative, hence the induction hypothesis yields an explicitly computable matrix $C \in GL(n-1,\mathbb{C})$ such that $C\tilde{A}_jC^{-1}$ is upper triangular. But then it is easy to see that the matrix $B = \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix} B_0$ is as desired.

In the real-spectrum case, all of the steps in the computation of B can be carried out over the reals. □

Proof of Theorem 4. We first compute bases of the subspaces in the decomposition

$$\mathbb{C}^n = \bigoplus_{\lambda \in \Lambda_{\mathbf{A}}} N(\mathbf{A}, \lambda)$$

of Lemma 7 (b). Since the spaces are A_j-invariant, for all j, it follows that A_j has block diagonal form, and the blocks of A_i, A_j associated to the same subspace commute. Furthermore, Lemma 7 (d) allows to choose real-valued bases whenever $\lambda \in \mathbb{R}^k$, and then the corresponding blocks are real-valued as well.

We let $\Lambda_{\mathbf{A}}^0$ denote the set of λ for which $N(\mathbf{A}, \lambda)$ is nontrivial. Pick a subset $\Lambda' \subset \Lambda_{\mathbf{A}}^0$ containing precisely one of $\lambda, \overline{\lambda}$, for each nonreal $\lambda \in \Lambda_{\mathbf{A}}^0$. It follows that

$$\mathbb{C}^n = \left(\bigoplus_{\lambda \in \Lambda_{\mathbf{A}} \cap \mathbb{R}^k} N(\mathbf{A}, \lambda) \right) \oplus \left(\bigoplus_{\lambda \in \Lambda'} N(\mathbf{A}, \lambda) \oplus N(\mathbf{A}, \overline{\lambda}) \right) .$$

For $\lambda \in \Lambda_{\mathbf{A}} \cap \mathbb{R}^k$, we compute a real-valued basis triangularizing the blocks of the A_j. Since each block has a single eigenvalue (this is the whole point of using the space $N(\mathbf{A}, \lambda)$), the triangularized blocks are therefore in $\mathbb{R} \cdot \mathbf{1}_d + \mathcal{N}(d, \mathbb{R})$, with d being the dimension of $N(\mathbf{A}, \lambda)$.

Finally, consider $\lambda \in \Lambda'$. Denote the block over $N(\mathbf{A}, \lambda)$ associated to A_j by B_j ($j = 1, \ldots, k$). Using Lemma 8, we can compute a basis $v_1, \ldots, v_d$ of $N(\mathbf{A}, \lambda)$ triangularizing the B_j. Hence there exist upper triangular matrices $C_j = (c^j_{s,r})_{s,r=1,\ldots,d}$ such that $B_j v_s = \sum_{r=1}^d c^j_{s,r} v_r$, for $s = 1, \ldots, d$. Since the B_j all have a single eigenvalue, the triangular matrices are in $\mathbb{C} \cdot \mathbf{1}_d + \mathcal{N}(d, \mathbb{C})$.

Let $V = N(\mathbf{A}, \lambda) \oplus N(\mathbf{A}, \overline{\lambda})$. Our next aim is to show that $\mathrm{Re}(v_1), \mathrm{Im}(v_1), \ldots, \mathrm{Re}(v_d), \mathrm{Im}(v_d)$ is a basis of V. For this purpose, first observe that $v \mapsto \overline{v}$ is a conjugate-linear bijective map between $N(\mathbf{A}, \lambda)$ and $N(\mathbf{A}, \overline{\lambda})$, which implies that $v_1, \ldots, v_d, \overline{v}_1, \ldots, \overline{v}_d$ is a basis of V. But this easily implies that $(\mathrm{Re}(v_1), \mathrm{Im}(v_1), \ldots, \mathrm{Re}(v_d), \mathrm{Im}(v_d))$ is a basis as well, and in addition real-valued. Now the fact that B_j is real-valued yields that

$$\begin{aligned} B_j \mathrm{Re}(v_s) = \mathrm{Re}(B_j v_s) &= \sum_{r=1}^d \mathrm{Re}(c^j_{s,r} v_r) \\ &= \sum_{r=1}^d \mathrm{Re}(c^j_{s,r}) \mathrm{Re}(v_r) - \mathrm{Im}(c^j_{s,r}) \mathrm{Im}(v_r) \ , \end{aligned}$$

and similarly

$$B_j \mathrm{Im}(v_s) = \sum_{r=1}^d \mathrm{Im}(c^j_{s,r}) \mathrm{Re}(v_r) + \mathrm{Re}(c^j_{s,r}) \mathrm{Im}(v_r) \ .$$

But that means that the matrix describing the restriction of B_j with respect to the basis $(\mathrm{Re}(v_1), \mathrm{Im}(v_1), \ldots, \mathrm{Re}(v_d), \mathrm{Im}(v_d))$ is given by $i_{\mathbb{C}}(C_j)$.

Thus, taking the properly indexed union of the bases constructed for each block yields a basis as postulated in Theorem 4. $\square$

We now turn to an application of Theorem 4. Recall that a necessary condition for admissibility of a matrix group is that it is closed. In the following, we will consider matrix groups of the form $H = \exp(\mathfrak{h})$, where $\mathfrak{h} \subset gl(n,\mathbb{R})$ is an abelian Lie-subalgebra. Such subalgebras are constructed by simply picking any set $A_1, \ldots, A_k$ of pairwise commuting, linearly independent matrices and letting $\mathfrak{h} = \mathrm{span}(A_1, \ldots, A_k)$. By construction, $H = \exp(\mathfrak{h})$ is a Lie-subgroup, and exp is a group epimorphism, when we consider $\mathfrak{h}$ with its additive group structure. However, it is not easy to decide whether H will be closed and/or simply connected. The remainder of this section is devoted to proving that these questions can be decided in a computational way, using the decomposition in Theorem 4.

We start out with a result setting closedness for the real spectrum case.

Lemma 9. *Let $\mathfrak{h} \subset gl(n,\mathbb{R})$ be an abelian Lie subalgebra with the property that* $\mathrm{spec}(X) \subset \mathbb{R}$ *for all $X \in \mathfrak{h}$. Let $H = \exp(\mathfrak{h})$ be the exponential image. Then H is a closed subgroup of $GL(n,\mathbb{R})$, and* $\exp : \mathfrak{h} \to H$ *is a diffeomorphism.*

Proof. Let $A_1, \ldots, A_k$ denote a basis of $\mathfrak{h}$. After choosing the right coordinates, we may assume (2) with B being the identity matrix. Since all A_i have real spectrum, it follows that $\mathbb{K}_r = \mathbb{R}$, for all r.

Now consider the Lie algebra $\mathfrak{g}$ generated by $A_1, \ldots, A_k$ and the block diagonal matrices with scalar multiples of the identity on the diagonal (with block sizes matching those of $A_1, \ldots, A_k$). By construction, $\mathfrak{g} = \mathfrak{d} + \mathfrak{n}$, where $\mathfrak{d}$ consists of diagonal matrices and $\mathfrak{n}$ consists of proper upper triangular matrices. Now the matrix exponential is a diffeomorphism of $\mathfrak{d}$ and $\mathfrak{n}$ onto closed matrix groups D and N, respectively (this is clear for $\mathfrak{d}$; for $\mathfrak{n}$, see [13, I.2.7]). By the assumptions on D and N, any element of D commutes with any element of N. Also, there is a smooth inverse of the embedding $D \to DN$, simply by setting the off-diagonal entries to zero. These facts imply that the canonical map $D \times N \to DN$ is a diffeomorphism, that DN is closed and that exp maps $\mathfrak{g}$ diffeomorphically onto DN.

But then $\mathfrak{h} \subset \mathfrak{g}$ is mapped diffeomorphically onto H, and H is closed as the image of the closed subspace $\mathfrak{h}$. $\square$

As it turns out, the closedness of an abelian matrix group will depend on the imaginary parts on the diagonal. The precise formulation of the conditions require some additional terminology, which is provided by the following lemma.

Lemma 10. *Let $\mathfrak{h} = \mathrm{span}(A_1, \ldots, A_k) \subset gl(n,\mathbb{R})$ be commutative. After an explicitly computable change of coordinates, there exist ℓ and $d_r, \mathbb{K}_r$, $r = 1, \ldots \ell$ such that*

$$\mathfrak{h} \subset \mathcal{A} = \left\{ \begin{pmatrix} B_1 & 0 & \ldots & 0 \\ 0 & B_2 & 0\ldots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \ldots & \ldots & B_\ell \end{pmatrix} : B_r \in \mathbb{K}_r \cdot \mathbf{1}_{d_r} + \mathcal{N}(d_r, \mathbb{K}_r) \right\} .$$

Then $\mathcal{A}$ is an associative subalgebra of $gl(n,\mathbb{R})$. Denote by $P_{\mathcal{E}} : \mathcal{A} \to \mathcal{A}$ the map that discards the imaginary parts on the diagonal, i.e.,

$$P_{\mathcal{E}} : \begin{pmatrix} \alpha_1 \cdot \mathbf{1}_{d_1} + N_1 & 0 & \ldots & 0 \\ 0 & \alpha_2 \cdot \mathbf{1}_{d_2} + N_2 & 0 \ldots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \ldots & \ldots & \alpha_\ell \cdot \mathbf{1}_{d_\ell} + N_\ell \end{pmatrix}$$
$$\mapsto \begin{pmatrix} \mathrm{Re}(\alpha_1) \cdot \mathbf{1}_{d_1} + N_1 & 0 & \ldots & 0 \\ 0 & \mathrm{Re}(\alpha_2) \cdot \mathbf{1}_{d_2} + N_2 & 0 \ldots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \ldots & \ldots & \mathrm{Re}(\alpha_\ell) \cdot \mathbf{1}_{d_\ell} + N_\ell \end{pmatrix} .$$

Let $\mathcal{E}$ denote the range of $P_{\mathcal{E}}$. Furthermore, let $P_{\mathcal{I}} = \mathrm{Id}_{\mathcal{A}} - P_{\mathcal{E}}$, and denote its range by $\mathcal{I}$. Then $\mathfrak{h}_0 = \mathfrak{h} \cap \mathcal{I}$ is explicitly computable, and there exists an explicitly computable complement $\mathfrak{h}_1$ of $\mathfrak{h}_0$ in $\mathfrak{h}$. On $\mathfrak{h}_1$, the map $P_{\mathcal{E}}$ is injective.

Proof. $\mathfrak{h}_0$ is the kernel of $P_{\mathcal{E}}|_{\mathfrak{h}}$, and thus one can compute a basis of this space, together with a basis of the complement, using the Gauss algorithm. Injectivity of $P_{\mathcal{E}}$ on the complement is clear. □

For the formulation of the next lemma, recall that an element g of a topological group G is called *a compact element* if the closed subgroup generated by g is compact. Given a sequence $(g_k)_{k\in\mathbb{N}} \subset G$, the statement $g_k \to \infty$ for $k \to \infty$ means that every compact subset $K \subset G$ contains at most finitely many g_k.

Lemma 11. *Let $\mathfrak{h} \subset gl(n,\mathbb{R})$ be commutative, and let $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ denote the decomposition from Lemma 10. Let $H_i = \exp(\mathfrak{h}_i)$.*

(a) *H_1 is closed and simply connected.*
(b) *$h \in H$ is a compact element of $GL(n,\mathbb{R})$ if and only if $h \in H_0$.*

Proof. First of all, note that $\exp(\mathcal{I}) \subset SO(n)$, the compact special orthogonal group, and thus in particular $\exp(\mathfrak{h}_0) \subset SO(n)$, which shows one direction of part (b), and will be crucial for part (a).

For the proof of part (a), we first prove that whenever $X_k \to \infty$ in $\mathfrak{h}_1$, then $\exp(X_k) \to \infty$ in $GL(n,\mathbb{R})$. For this purpose, let $\mathfrak{h}_2 = P_{\mathcal{E}}(\mathfrak{h}_1)$ and $H_2 = \exp(\mathfrak{h}_2)$. Then $X_k = Y_k + Z_k$, with $Y_k \in \mathcal{I}$ and $Z_k = P_{\mathcal{E}}(X_k) \in \mathfrak{h}_2$. By injectivity of $P_{\mathcal{E}}$ on $\mathfrak{h}_1$, it follows that $Z_k \to \infty$. By Lemma 9, $\exp : \mathfrak{h}_2 \to H_2$ is a diffeomorphism onto the closed subgroup H_2, and thus $\exp(Z_k) \to \infty$ in $GL(n,\mathbb{R})$. Hence, if $K \subset GL(n,\mathbb{R})$ is compact, then so is $K' = SO(n)K$, and it can contain only finitely many $\exp(Z_k)$. On the other hand, $\exp(X_k) \in K$ implies $\exp(Z_k) = \exp(-Y_k)\exp(X_k) \in K'$, thus K contains only finitely many $\exp(X_k)$. Hence $\exp(X_k) \to \infty$. This implies in particular that the kernel of exp must be compact, hence trivial. So H_1 is simply connected.

Furthermore, it follows that H_1 is closed: Assume that $\exp(X_k) \to g \in GL(n,\mathbb{R})$. This implies that the sequence $(X_k)_{k\in\mathbb{N}}$ does not converge to infinity, hence it contains a bounded, and thus finally a convergent subsequence $X_{n_k} \to X_0$, and $X_0 \in \mathfrak{h}_1$ since linear subspaces are closed. But then continuity of exp yields $\exp(X_0) = g$, and thus $g \in H_1$.

For the missing direction of part (b), write $h = \exp(X_0 + X_1)$, with $X_i \in \mathfrak{h}_i$. If $X_1 \neq 0$, the argument proving part (a) shows that $h^k = \exp(k(X_0 + X_1)) \to \infty$. In particular, h is not a compact element. □

The following theorem reveals the chief purpose of the introduction of $\mathfrak{h}_0, \mathfrak{h}_1$: Closedness only depends on $\mathfrak{h}_0$.

Theorem 12. *Let $\mathfrak{h} \subset gl(n,\mathbb{R})$ be commutative, and let $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ denote the decomposition from Lemma 10. Then $H = \exp(\mathfrak{h})$ is closed if and only if $H_0 = \exp(\mathfrak{h}_0)$ is compact.*

Proof. By Lemma 11, H_1 is closed. Thus, if H_0 is compact, then H is the product of a closed and a compact subset of $GL(n,\mathbb{R})$, hence closed. On the other hand, if H is closed, then its subgroup of compact elements is closed as well, and compactness of an element relative to H is the same as compactness relative to $GL(n,\mathbb{R})$. This subgroup coincides with H_0 by Lemma 11, and it is closed by [12, Theorem 9.10]. In summary: H_0 is closed, hence compact, since it is contained in $SO(n)$. □

Hence, we need methods to identify closed subgroups of the torus group $\mathbb{T}^d = \{z \in \mathbb{C}^d : |z_1| = |z_2| = \ldots = |z_d| = 1\}$. In the following, we identify the Lie algebra of $\mathbb{T}^d$ with $\mathbb{R}^d$, and $\exp : \mathbb{R}^d \to \mathbb{T}^d$ is given by $\exp(x_1,\ldots,x_d) = (e^{ix_1},\ldots,e^{ix_d})$.

Lemma 13. *Let $\varphi = (\varphi_1,\ldots,\varphi_d) \in \mathbb{R}^d$, and denote by H the closure of* $\exp(\mathbb{R}\varphi)$. *Let $\mathfrak{h} \subset \mathbb{R}^d$ denote the Lie algebra of H, i.e.*

$$\mathfrak{h} = \{x \in \mathbb{R}^d : \exp(x) \in H\} \ .$$

Suppose that $1 \le i_0 \le d$ denotes the smallest index of an irrational entry of φ. Then $\mathfrak{h}$ contains a vector whose first nonzero component is at position i_0.

Proof. Since replacing φ by any nonzero scalar multiple yields the same one-parameter group, we may assume that the first $i_0 - 1$ entries are integers. Denote by H_0 the closure of the cyclic subgroup $\{\exp(2\pi k\varphi) : k \in \mathbb{Z}\}\subset \mathbb{T}^d$. Then, if $p : \mathbb{T}^d \to \mathbb{T}^{i_0}$ denotes the projection onto the first i_0 components, we claim that $p(H_0) = \{1\}^{i_0-1} \times \mathbb{T}$. Indeed, by choice of φ, i_0 and H_0 we have

$$\{1\}^{i_0-1} \times \{e^{2\pi i k\varphi_{i_0}} : k \in \mathbb{Z}\} \subset p(H_0) \subset \{1\}^{i_0-1} \times \mathbb{T} \ . \tag{3}$$

Furthermore, $p(H_0)$ is the continuous image of a compact set, and thus closed. By choice of i_0, $\{e^{2\pi i k\varphi_{i_0}} : k \in \mathbb{Z}\} \subset \mathbb{T}$ is dense. Hence the first inclusion of (3) and closedness of $p(H_0)$ implies $p(H_0) = \{1\}^{i_0-1} \times \mathbb{T}$.

In particular, H_0 is a closed infinite subgroup of $\mathbb{T}^d$, and therefore it is a Lie subgroup of positive dimension. If $\mathfrak{h}_0$ denotes its Lie algebra, then (3) implies that $\mathfrak{h}_0 \subset \{0\}^{i_0-1} \times \mathbb{R}^{d+1-i_0}$. On the other hand, $\mathfrak{h}_0 \not\subset \{0\}^{i_0} \times \mathbb{R}^{d-i_0}$, since $p(H_0)$ is nontrivial. But this shows the statement. □

With this lemma, closedness of a Lie-subgroup of $\mathbb{T}^d$ is easily decided. We first recall some notions connected to Gauss elimination: Let a family of vectors $v_j = (v_j(1), \ldots, v_j(d)) \in \mathbb{R}^d$, $j = 1, \ldots, k$ be given. We say that the vectors are in *Gauss-Jordan row echelon form* if there exist indices $1 \leq i_1 < i_2 < \ldots < i_j \leq d$ such that the following properties hold, for all $j = 1, \ldots, k$:

$$v_j(r) = 0 \ , \quad \text{for } r < i_j \ , \ v_j(i_j) = 1 \ , \text{ and } v_j(i_\ell) = 0 \ , \quad \text{for } \ell \neq j \ .$$

For any finite family of vectors, a system of vectors in Gauss-Jordan row echelon form spanning the same space can be computed by first computing the echelon form using Gauss elimination, normalizing the resulting vectors to have unit pivot elements, and using the pivot element of each vector to eliminate the corresponding entries in the other vectors.

Lemma 14. *Let $v_1, \ldots, v_k \in \mathbb{R}^d$ be given in Gauss-Jordan row echelon form, and let $\mathfrak{h} = \mathrm{span}(v_j : j \leq k)$. Then $H = \exp(\mathfrak{h})$ is compact iff $v_j \in \mathbb{Q}^d$, for all $1 \leq j \leq k$.*

Proof. First assume that $v_j \in \mathbb{Q}^d$. Then $\exp(2\pi k v_j) = (1, \ldots, 1)$ for a suitable integer $k > 0$, showing that $\exp(\mathbb{R}v_j)$ is compact. If this holds for all j, then H is compact.

Conversely, assume that H is compact, but some v_j has an irrational entry. Let ℓ be the smallest index with $v_j(\ell) \notin \mathbb{Q}$. Then, by the previous lemma, $\mathfrak{h}$ contains a vector φ whose first nonzero component is at position ℓ. On the other hand, since $v_j(\ell) \neq 0$, the fact that the vectors are in Gauss-Jordan row echelon form implies that $\ell \notin \{i_j : j = 1, \ldots, k\}$. But clearly, that is a contradiction to $\varphi \in \mathfrak{h}$. □

To summarize:

Theorem 15. *Let $\mathfrak{h} \subset gl(n, \mathbb{R})$ be an abelian Lie algebra. Assume that for some system of generators of $\mathfrak{h}$ the spectra are known. Then the question whether $H = \exp(\mathfrak{h})$ is closed can be decided by repeated applications of Gauss elimination.*

Proof. First compute the decomposition in Theorem 4. From this decomposition, determine the direct sum decomposition $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ from Lemma 10. Then compute a basis of $\mathfrak{h}_0$ in Gauss-Jordan echelon form and check for irrational entries. □

4. From discrete to continuous admissible matrix groups

A full classification of admissible abelian matrix groups, say up to conjugacy, does not seem feasible. As the discussion in [8] shows, already the problem of classifying, up to conjugacy, all connected abelian matrix groups with open orbits is equivalent to the classification of commutative algebras with unity of the same dimension. For the latter

problem, no solution is in sight. Moreover, we are interested also in non-connected, even discrete abelian groups, and here the diversity is even larger. In view of this fact, it seems reasonable to first restrict attention to suitable subclasses, which are particularly easy to handle, and then to find ways how to pass from the subclass to general abelian matrix groups. For the first part, connected groups seem particularly well suited, since the exponential map allows to systematically translate group-theoretical questions to problems in linear algebra. For the passage to other closed abelian matrix groups, the role of cocompactness will be crucial. The following simple lemma provides the key.

Lemma 16. *Let H_0, H_1 be closed abelian matrix groups, with $H_0 \subset H_1$ cocompact, i.e. H_1/H_0 is compact. Then H_0 is admissible iff H_1 has the same property.*

Proof. First suppose that H_0 is admissible. Let $\psi \in \mathrm{L}^2(\mathbb{R}^n)$ be an admissible vector, and define $G_i = \mathbb{R}^n \rtimes H_i$. Pick a measurable fundamental domain C of H_1/H_0, and endow it with the Haar measure of H_1/H_0. Then, with suitable normalizations, we have for $f \in \mathrm{L}^2(\mathbb{R}^n)$ that

$$\begin{aligned}
\|V_\psi(f)\|^2_{\mathrm{L}^2(G_1)} &= \int_{H_1}\int_{\mathbb{R}^n} |\langle f, \pi(h,x)\psi\rangle|^2 |\det(h)|^{-1} dx \frac{dh}{|\det h|} \\
&= \int_C \int_{H_0}\int_{\mathbb{R}^n} |\langle f, \pi(ch_0,x)\psi\rangle|^2 dx \frac{dh_0}{|\det(h_0)|}\frac{dc}{|\det c|} \\
&= \int_C \|V_\psi(\pi(c^{-1})f)\|^2_{\mathrm{L}^2(G_0)} \frac{dc}{|\det c|} \\
&= \|f\|_2^2 \int_C \frac{dc}{|\det c|} .
\end{aligned}$$

Hence ψ is H_1-admissible up to normalization.

For the converse, let ψ_1 denote an H_1-admissible function. Using the measurable fundamental domain C from above, we define

$$\widehat{\psi_0}(x) = \left(\int_C |\widehat{\psi_1}(c^{-1}x)|^2 dc \right)^{1/2} .$$

Then a straightforward computation (using that H_1 is commutative) shows that $\psi_0 \in \mathrm{L}^2(\mathbb{R}^n)$, and that it fulfills the Calderon condition (1) for H_0. □

The lemma motivates the following definition: Given $H_1, H_0 \subset \mathrm{GL}(n,\mathbb{R})$, we write $H_0 \sim H_1$ iff $H_0 \cap H_1$ is cocompact in both H_0 and H_1. Then the previous lemma implies that this relation is compatible with admissibility: If $H_0 \sim H_1$, then H_0 is admissible iff H_1 is. Furthermore, $\sim$ is reflexive and symmetric, but not transitive. We therefore introduce its transitive hull, denoted by $\approx$; that is, $H_0 \approx H_1$ means that H_0, H_1 can be connected by a chain of groups related by $\sim$. This is an equivalence relation which is compatible with admissibility.

We now employ structure theory of compactly generated LCA groups to prove the following proposition, which shows that each equivalence class modulo $\approx$ contains a connected,

simply connected representative H_c. Thus, in principle, the discussion may be restricted to this subclass. Note however that the construction of H_c is not particularly explicit, and thus of somewhat limited use for the discussion of concrete examples. The subsequent remark 19 provides a more direct construction of H_c for the subclass of discrete matrix groups with positive spectrum.

Proposition 17. *Let $H < \mathrm{GL}(n,\mathbb{R})$ be closed and abelian.*

(a) *$H \cong \mathbb{R}^l \times \mathbb{Z}^m \times \mathbb{T}^k \times F$, with $l,m,k \in \mathbb{N}_0$, and F denotes a finite abelian group. The isomorphism is topological.*
(b) *There exists $H_c \sim H$ with $H_c \cong \mathbb{R}^j$, and $j = l+m$. Moreover, there exists $H_d \subset H \cap H_c$, cocompact in both, with $H_d \cong \mathbb{Z}^j$. As a consequence of $H_c \sim H \sim H_d$, H is admissible iff H_d is.*

Proof. For part (a) confer [21]. The result basically follows from the fact that closed abelian subgroups of $GL(n,\mathbb{R})$ are compactly generated, and a structure theorem for such groups contained in [12].

For the construction of H_c, we first get rid of the compact part of H, i.e., we let H_0 denote the subgroup corresponding to $\mathbb{R}^l \times \mathbb{Z}^m$. The remaining problem consists therefore in suitably embedding the discrete part into a vector group.

For this purpose we let $\mathcal{A} = \mathrm{span}(H_0)$, the matrix algebra generated by H_0. Then $H_0 \subset \mathcal{A}^\times$, where the latter denotes the group of invertible elements in $\mathcal{A}$. We claim that $\mathcal{A}^\times = \mathcal{A} \cap GL(n,\mathbb{R})$, whence is a closed subgroup of $GL(n,\mathbb{R})$ (because the algebra $\mathcal{A}$ is a linear subspace, hence closed). Here the inclusion $\subset$ is clear. For the other direction suppose that a matrix $X \in \mathcal{A}$ is invertible. Then left multiplication with X is an injective linear map of $\mathcal{A}$ into itself, thus it is also onto. Hence the (right) inverse of X is also in $\mathcal{A}$. The group $\mathcal{A}^\times$ is also almost connected by [8, Proposition 10]. Hence the structure theorem for compactly generated LCA groups yields $\mathcal{A}^\times \cong \mathbb{R}^s \times \mathbb{T}^t \times F$, with a finite abelian group F. Possibly by replacing H_0 by a closed subgroup of finite index, we may assume that $\pi_F(H_0)$ is trivial, where π_F is the projection map onto F. Indeed, we have $H_0 \subset \mathcal{A}^\times \cong \mathbb{R}^s \times \mathbb{T}^t \times F$, and π_F is a continuous homomorphism. Then $\pi_F(H_0) \subset F$ is a finite group, which means that the kernel K of π_F, restricted to H_0, is a closed subgroup of finite index: The isomorphism theorem states that $\pi_F(H_0) \simeq H_0/K$. We replace H_0 by K.

For the sake of explicitness, we introduce topological isomorphisms $\phi : \mathbb{R}^l \times \mathbb{Z}^m \to H_0$ and $\psi : \mathcal{A}^\times \to \mathbb{R}^s \times \mathbb{T}^t \times F$. Let $\psi_1, \ldots, \psi_{s+t}$ denote the $\mathbb{R}^s \times \mathbb{T}^t$-valued components of ψ. Then

$$\Theta : \mathbb{R}^l \times \mathbb{Z}^m \ni (x,m) \mapsto (\psi_1(\phi(x,m)), \ldots, \psi_{s+t}(\phi(x,m))) \in \mathbb{R}^s \times \mathbb{T}^t$$

is a continuous group monomorphism, and it is topological onto its image. This image is $\psi(H_0)$, whence closed as H_0 is closed in $\mathcal{A}^\times$. Our aim is to extend Θ to a continuous monomorphism $\tilde{\Theta} : \mathbb{R}^{l+m} \to \mathbb{R}^s \times \mathbb{T}^t$.

First observe that already the map $\Theta_0 : \mathbb{R}^l \times \mathbb{Z}^m \ni (x,m) \mapsto (\psi_1(\phi(x,m)),\ldots,\psi_s(\phi(x,m))) \in \mathbb{R}^s$ is injective: The kernel of Θ_0 is mapped by Θ onto a closed subgroup of $\{0\} \times \mathbb{T}^t$. Hence it is compact, and thus trivial.

Since $\mathbb{T}^t$ is compact, the projection $\mathbb{R}^s \times \mathbb{T}^t \to \mathbb{R}^s$ is a closed mapping, and thus Θ_0 has closed image also. Hence [12, Theorem 9.12] applies to yield that

$$\{\Theta_0(1,0,\ldots,0),\ldots,\Theta_0(0,\ldots,0,1)\} \subset \mathbb{R}^s$$

is $\mathbb{R}$-linearly independent, giving rise to a linear monomorphism $\mathbb{R}^{l+m} \to \mathbb{R}^s$, written as a tuple $(\tilde{\theta}_1,\ldots,\tilde{\theta}_s)$ of homomorphisms $\mathbb{R}^{l+m} \to \mathbb{R}$.

Now pick $z_{i,j} \in \mathbb{R}$ $(1 \le i \le m, s+1 \le j \le s+t)$ such that $\psi_j(\phi(\delta_i)) \in (z_{i,j} + \mathbb{Z}^t)$, where $\delta_i \in \mathbb{R}^l \times \mathbb{Z}^m$ denotes the vector with 1 as $l+i$ th entry, and zeros elsewhere. Letting

$$\tilde{\theta}_j(x,y) = \psi_j(\phi(x,0)) + (\sum_{i=1}^m y_i z_{i,j} + \mathbb{Z}^t) \ , \qquad j = s+1,\ldots,s+t$$

for $(x,y) \in \mathbb{R}^l \times \mathbb{R}^m$ therefore gives rise to an extension

$$\widetilde{\Theta} : \mathbb{R}^l \times \mathbb{R}^m \ni (x,y) \mapsto (\tilde{\theta}_1(x,y),\ldots,\tilde{\theta}_{s+t}(x,y)) \in \mathbb{R}^s \times \mathbb{T}^t$$

of Θ; then $\tilde{\Theta}(\mathbb{R}^{l+m})$ is closed in $\mathbb{R}^s \times \mathbb{T}^t$, since already the projection onto the first s components yields the closed subgroup $\Theta_0(\mathbb{R}^\ell \times \mathbb{Z}^m)$.

Now $H_c = (\psi^{-1} \circ \widetilde{\Theta})(\mathbb{R}^{l+m})$ is as desired: by construction of $\widetilde{\Theta}$, $H_0 = (\psi^{-1} \circ \widetilde{\Theta})(\mathbb{R}^l \times \mathbb{Z}^m)$, with $H_c/H_0 \cong \mathbb{T}^m$. Finally, $H_d = (\psi^{-1} \circ \widetilde{\Theta})(\mathbb{Z}^{l+m})$ is discrete and cocompact in H_0 and thus in H. □

Both the structure of the group and the geometrical intuition of the action simplify greatly if we can assume that all matrices in the group have real eigenvalues.

Definition 18. An abelian matrix group H **has real (positive) spectrum** if all $h \in H$ have only real (positive) eigenvalues. □

Remark 19. (a) We note that the group H_c in Proposition 17 (b) can possibly be explicitly computed: Let $B_1,\ldots,B_l$ denote infinitesimal generators of the subgroup of H corresponding to $\mathbb{R}^l$, $B_{l+1},\ldots,B_{l+m}$ generators of the $\mathbb{Z}^m$ part, $B_{l+m+1},\ldots,B_{l+m+k}$ infinitesimal generators of the $\mathbb{T}^k$ part, and $B_{l+k+m+1},\ldots,B_{l+k+m+f}$ the elements of the finite group.

Then these matrices commute: This is clear for any pair of matrices contained in the discrete part. Moreover, if $1 \le i \le l$ and $l+1 \le j \le l+m$, differentiating the equality $[\exp(rB_i), B_j] = 0$ and evaluating at $r = 0$ yields $[B_i, B_j] = 0$. Similar arguments apply to the remaining cases, showing that all matrices in the list $B_1,\ldots,B_{l+k+m+f}$ commute, and can therefore be jointly decomposed into block triangular form as described in Theorem 4. But this block structure is preserved by the exponential map, hence it follows that all elements of H have the same block structure.

Thus the decomposition of Theorem 4 is valid for the group H as well. Now the explicit construction of H_c depends on the unit group of the matrix algebra $\mathcal{A} = \mathrm{span}(H)$, which is computable from the decomposition of the generators into blocks.

(b) The observation in part (a) simplifies the reasoning in particular for the case of positive spectrum, to the extent that H_c can be computed explicitly: Suppose that H has only positive eigenvalues. Here, the logarithms of the generators can be computed using the power series

$$\log(A) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(A - \mathbf{1})^k \ . \tag{4}$$

First assume that we only have one block. Then A is of the form $a \cdot (\mathbf{1} + N)$, with a a positive real and N nilpotent, and we can derive from this

$$\log(A) = \log(a) \cdot \mathbf{1} + \sum_{k=1}^{s-1} \frac{(-1)^{k+1}}{k} N^k \ , \tag{5}$$

where s is the block size. While these calculations are somewhat informal, it can be shown by direct calculation that by defining $\log(A)$ as in (5), we indeed have $\exp(\log(A)) = A$ [13]. This procedure is applied blockwise to yield the logarithm for the general case.

With the notations from part (a), we let $\mathfrak{h}_c = \mathrm{span}\,(B_1, \ldots, B_l, \log(B_{l+1}), \ldots, \log(B_{l+m}))$; noting that the logarithms are computed in finitely many steps. We claim that $H_c = \exp(\mathfrak{h}_c)$ is as desired: It is closed and simply connected by Lemma 9, and $\exp : \mathfrak{h}_c \to H_c$ is a diffeomorphism. Furthermore, the additive quotient group $\mathfrak{h}_c / \langle B_1, \ldots, \log(B_{l+m}) \rangle$ is quasicompact, by construction of $\mathfrak{h}_c$. Then the same is true of the exponential images. But this implies that H_c/H_d is quasicompact, thus compact. □

Remark 20. Let $H < \mathrm{GL}(n, \mathbb{R})$ be a closed abelian subgroup with real spectrum. Then H contains a closed cocompact subgroup with positive spectrum. To see this, assume that $\theta : H \to \mathbb{R}^l \times \mathbb{Z}^m \times \mathbb{T}^k \times F$ (F finite) is a topological isomorphism, and let $H_0 = \{h^2 : h \in H\}$. Then H_0 is a subgroup with positive spectrum, with $\theta(H_0) = \mathbb{R}^l \times (2\mathbb{Z})^m \times \mathbb{T}^k \times F'$, where F' is the subgroup of squares in F. Thus H_0 is as desired. □

5. Admissibility for closed abelian simply connected matrix groups

Proposition 17 leads us to consider the admissibility of a closed abelian simply connected matrix group. In this section we will study this question and give some generalizations at the end. The main results are Theorem 23 and its generalization, Theorem 30.

The following necessary condition can be immediately derived from Proposition 17: If $H \cong \mathbb{R}^l \times \mathbb{Z}^m \times \mathbb{T}^k \times F$, and $H \sim H_c$, with $H_c \cong \mathbb{R}^{l+m}$, then a necessary condition for H to be admissible is that H_c acts freely almost everywhere. The reason is that H_c is admissible, and thus almost every stabilizer in H_c is compact, hence trivial.

Since one motivation for introducing connected groups to the discussion is their accessibility via Lie algebras, it is therefore natural to consider Lie algebra criteria for the property

that H_c acts freely. Note that this usually only contains information about *local* mapping properties of the group action, hence we can at best obtain a criterion for the stabilizers to be discrete. This is easily seen to be equivalent to *local freeness* of the action: H acts locally freely on $H.x$ if there exists a neighborhood U of unity in H such that $h \mapsto h.x$ is injective on U. The following proposition studies the characterization of locally free actions.

Proposition 21. *Let $H < GL(n, \mathbb{R})$ be a Lie subgroup of dimension l, with Lie algebra $\mathfrak{h} \subset gl(n, \mathbb{R})$.*

(a) *For $x \in \mathbb{R}^n$, the stabilizer $\mathrm{stab}_H(x)$ is discrete iff*

$$\text{The linear map } \mathfrak{h} \ni A \mapsto Ax \in \mathbb{R}^n \text{ has rank } l\,. \tag{6}$$

Hence, necessarily $l \leq n$.

(b) *If $\mathrm{stab}_H(x)$ is discrete for one $x \in \mathbb{R}^n$, then there exists an open H-invariant set $U \subset \mathbb{R}^n$ such that $|\mathbb{R}^n \setminus U| = 0$, and $\mathrm{stab}_H(y)$ is discrete for all $y \in U$.*

(c) *Given a basis of $\mathfrak{h}$, the existence of $x \in \mathbb{R}^n$ with discrete stabilizer can be checked computationally.*

Proof. The stabilizer is discrete iff the canonical mapping $H \ni h \mapsto hx$ is a local homeomorphism onto the orbit $H.x$. But the mapping from (6) is the derivative of this map at unity. This proves part (a).

For part (b) we note that (6) is fulfilled iff there exists an index set $I \subset \{1, \ldots, n\}$ with $|I| = l$, such that the mapping

$$M_{I,x} : \mathfrak{h} \ni A \mapsto ((Ax)_i)_{i \in I} \in \mathbb{R}^I$$

is a vector space isomorphism. This is the case iff $\det(M_{I,x}) \neq 0$. We let $P_I(x) = \det(M_{I,x})$, and observe that since the determinant is polynomial, and x enters linearly in the definition of $M_{I,x}$, $P_I(x)$ is indeed a polynomial.

It follows that the set of all x for which (6) holds is characterized by the condition $P(x) \neq 0$, where

$$P(x) = \sum_{I \subset \{1,\ldots,n\}, |I| = l} P_I^2(x)\ .$$

This set is clearly open, and if it is not empty, its complement has measure zero. This settles (b), and the algorithm for (c) is now clear: Check whether all coefficients of P vanish. Since the degree of P is $\leq 2l$, this can be done in finitely many steps. □

As a further necessary criterion, we note:

Corollary 22. *Let $H < GL(n, \mathbb{R})$ be an admissible abelian matrix group, $H \cong \mathbb{R}^l \times \mathbb{Z}^m \times \mathbb{T}^k \times F$. Then $l + m \leq n$.*

Theorem 23. *Let $H < \mathrm{GL}(n, \mathbb{R})$ be a closed simply connected abelian matrix group of dimension k with positive spectrum. Then H is admissible if and only if*

(i) *There exists* $h \in H$ *with* $|\det(h)| \neq 1$*;*
(ii) *For some* $x \in \mathbb{R}^n$ *(and hence for almost all) the linear map*

$$\mathfrak{h} \ni A \mapsto xA \in \mathbb{R}^n \tag{7}$$

has rank k*.*

Proof. Condition (i) is necessary by Theorem 2, and necessity of condition (ii) has been shown at the beginning of the section. For the sufficiency we note first that by the above considerations condition (ii) holds if and only if there exists at least one (and hence almost all) dual orbit on which H acts locally freely. We will proceed by induction over the number of blocks in the joint tridiagonalization of the infinitesimal generators of H.

The following theorem, which is mainly due to Chevalley and Rosenlicht, is essential for treating the single block case. Note that the theorem yields a measurable set meeting *every* orbit, not just almost every orbit, in a single point.

Theorem 24. *Let* H *be a closed connected subgroup of* $T(n,\mathbb{R})$*, the group of upper triangular matrices with ones on the diagonal. Then there exists a measurable subset of* $\mathbb{R}^n$ *meeting each* H*-orbit in precisely one point. Moreover,* H *acts freely on each orbit on which it acts locally freely.*

Proof. We rely on a theorem relating the existence of measurable fundamental domains to regularity of the orbits. More precisely, suppose a locally compact group H acts continuously on the locally compact space $\mathbb{R}^n$. Then by a result of Effros [6, Theorems 2.1–2.9, (2) $\Leftrightarrow$ (12)], there exists a measurable set meeting each orbit in $\mathbb{R}^n$ precisely once iff for every $x \in U$ the natural mapping $H \in h \mapsto h.x$ induces a homeomorphism $H/\mathrm{stab}_H(x) \to H.x$. But the Chevalley-Rosenlicht theorem [5, Theorem 3.1.4] shows precisely that, with the additional information that the H-orbits are simply connected.

The second fact then implies that H acts freely whenever it acts locally freely: If $\mathrm{stab}_H(x)$ is discrete, it follows that $\mathrm{stab}_H(x) \cong \mathbb{Z}^m$, and then $H/\mathrm{stab}_H(x) \cong \mathbb{R}^{n-m} \times \mathbb{T}^m$. But the quotient is simply connected, which implies that $m = 0$. □

The following lemma is needed for the induction step.

Lemma 25. *Let* $H_i < GL(V_i)$ *be given, where* V_i *are vector spaces. Let* $\sigma : H_1 \to GL(V_2)$ *be a group homomorphism with* $\sigma(h_1)h_2 = h_2\sigma(h_1)$*, for all* $h_1 \in H_1, h_2 \in H_2$*. Let* $H < GL(V_1 \oplus V_2)$ *be given by*

$$H = \{h = (h_1, \sigma(h_1)h_2) : h_1 \in H_1, h_2 \in H_2\} \ .$$

Let $S_i \subset V_i$ *denote a measurable fundamental domain of* V_i/H_i*, with the additional property that* H_i *acts freely on the orbits running through* S_i*. Then* $S_1 \times S_2$ *is a measurable fundamental domain for* $(V_1 \oplus V_2)/H$*, and* H *acts freely on the orbits running through* S*.*

Proof. We compute

$$U = H.(S_1 \times S_2) = \{(h_1.s_1, \sigma(h_1)h_2.s_2) : h_i \in H_i, s_i \in S_i\} \ .$$

Then for almost every $y \in V_1$, $y = h_1.s_i$ for suitable $h_1 \in H_1$ and $s_1 \in S_1$. Moreover, for each such y, the slice

$$U \cap \{y\} \times V_2$$

contains the elements $\sigma(h_1)h_2.s_2$, where $h_2 \in H_2$ and s_2 in S. Since $\sigma(h_1)$ is a fixed invertible linear mapping, the slice has complement of measure zero in $\{y\} \times V_2$ (in the Lebesgue measure on the affine subspace). Hence Fubini's theorem implies $|V_1 \times V_2 \setminus U| = 0$.

For the second property of a measurable fundamental domain, assume $(s_1, s_2) = h.(\tilde{s}_1, \tilde{s}_2) = (h_1\tilde{s}_1, \sigma(h_1)h_2.\tilde{s}_2)$, for suitable $s_i, \tilde{s}_i \in S_i$ and $h = (h_1, h_2) \in H$. Then the fact that S_1 is a fundamental domain yields that $s_1 = \tilde{s}_1$, and h_1 is contained in the stabilizer. By the freeness assumption, $h_1 = \mathbf{1}$. Hence $h_2\tilde{s}_2 = s_2$, resulting again in $s_2 = \tilde{s}_2$ and $h_2 = \mathbf{1}$.

We remark that this argument also covers the special case that H_2 is trivial; in this case, $S_2 = V_2$. □

Now let $A_1, \ldots, A_k$ denote a basis of the Lie algebra of H. We assume that H has positive eigenvalues, hence all eigenvalues of the A_i are real. Therefore there exists a basis of $\mathbb{R}^n$ with respect to which the A_i have the block diagonal form from Theorem 4. The proof now proceeds by induction over the number of blocks in the decomposition.

First suppose that the number of blocks equals one, i.e. each basis element has a single eigenvalue. If this eigenvalue is zero, for all A_i, then $H < T(n, \mathbb{R})$, and Theorem 24 applies. In the remaining case, we may assume after replacing some of the A_i by suitable linear combinations, that A_1 has eigenvalue 1, and the remaining A_i have eigenvalue 0. Hence $A_1 = \mathbf{1} + N$, with a suitable proper upper tridiagonal matrix N.

Hence $H = H_1H_2$, where $H_1 = \exp(\mathbb{R}A_1)$, and $H_2 = \exp(\mathrm{span}(A_i : i = 2, \ldots, k))$. Note that $H_2 < T(n, \mathbb{R})$. Hence by theorem 24, there exists a measurable set S_2 meeting each H_2-orbit in precisely one point. Observe that for all $h_2 \in H_2$ and all $x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n$, $h_2(x) = (y, x_n)$, for suitable $y \in \mathbb{R}^{n-1}$. In particular, the affine subspaces $\mathbb{R}^{n-1} \times \{\pm 1\}$ are H_2-invariant. Hence, $S = S_2 \cap (\mathbb{R}^{n-1} \times \{\pm 1\})$ is a measurable set meeting each H_2-orbit in this H_2-invariant subset precisely once.

We claim that S meets each orbit in $\mathbb{R}^{n-1} \times (\mathbb{R} \setminus \{0\})$ in precisely one point. To see this, observe that $h_1 = \exp(rA_1)$ factors as $h_1(x) = e^r u(r)$, with $u(r)$ a unipotent matrix, keeping the nth coordinate of each vector fixed.

Let (y, x_n) with $y \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R} \setminus \{0\}$ be given. Let $r = \log(|x_n|)$. Then $\exp(-rA_1)(y, x_n) = (y', \mathrm{sign}(x_n))$, with suitable $y' \in \mathbb{R}^{n-1}$. By choice of S there exists $s \in S$ and unique $h \in H_2$ such that $(y, \mathrm{sign}(x_n)) = h.s$. This shows that each orbit is met.

Moreover, if $h_1h_2.s = \tilde{s}$ for $s, \tilde{s} \in S$ and $h_i \in H_i$, the comparison of the nth coordinates shows that h_1 is the identity. But $h_2.s = \tilde{s}$ implies $s = \tilde{s}$, because of $S \subset S_2$.

This shows that there exists a measurable set meeting each orbit in $\mathbb{R}^{n-1} \times (\mathbb{R} \setminus \{0\})$ precisely once. The action is free wherever it is locally free: For H_2, this observation

was part of Theorem 24, for H_1H_2 it follows from this and the fact that H_1 acts freely on the nth variable. Hence we may replace S by the smaller set $S' \subset S$ of all points in S fulfilling in addition the local freeness condition (6). This condition determines a measurable subset S' of S, and the orbits going through this subset yield the intersection of $\mathbb{R}^{n-1} \times (\mathbb{R} \setminus \{0\})$ with the set of all orbits of maximal dimension. By Proposition 21 and the hypothesis of the theorem, this is a Borel set with complement of measure zero. Hence we have finally produced a measurable fundamental domain S' with the property that H acts freely on the orbits through S'. This concludes the one block case.

Now suppose we have shown the statement for ℓ blocks, and assume that the joint block diagonalization of the Lie algebra basis for H has $\ell+1$ blocks. Let k' denote the dimension of the space spanned by the matrices $\tilde{A}_1, \ldots, \tilde{A}_k$, obtained by taking the first ℓ blocks of $A_1, \ldots, A_k$. Denote by $n' < n$ the sum of the sizes of the first ℓ blocks.

Then, passing to suitable linear combinations and reindexing allows the assumption that $\tilde{A}_{k'+1} = \ldots = \tilde{A}_k = 0$. We let $H_1^s = \exp(\mathrm{span}(\tilde{A}_1, \ldots, \tilde{A}_{k'})) \subset \mathrm{GL}(n', \mathbb{R})$, $H_2^s \subset \mathrm{GL}(n-n', \mathbb{R})$ the complementary subgroup corresponding to the remaining basis elements, $H_1^b = \exp(\mathrm{span}(A_1, \ldots, A_{k'})) \subset \mathrm{GL}(n, \mathbb{R})$, and $H_2^b = \exp(\mathrm{span}(A_{k'+1}, \ldots, A_k)) \subset \mathrm{GL}(n, \mathbb{R})$. Then there exists a group homomorphism $\sigma : H_1^s \to \mathrm{GL}(n-n', \mathbb{R})$ such that

$$H_1^b = \left\{ \begin{pmatrix} h_1 & 0 \\ 0 & \sigma(h_1) \end{pmatrix} : h_1 \in H_1^s \right\} , \tag{8}$$

and

$$H = H_1^b H_2^b = \left\{ \begin{pmatrix} h_1 & 0 \\ 0 & \sigma(h_1)h_2 \end{pmatrix} : h_1 \in H_1^s, h_2 \in H_2^s \right\} . \tag{9}$$

The induction hypothesis yields measurable fundamental domains S_1 and S_2 for $\mathbb{R}^{n'}/H_1^s$ and $\mathbb{R}^{n-n'}/H_2^s$, respectively. Then by Lemma 25, $S = S_1 \times S_2$ is a measurable fundamental domain for $\mathbb{R}^n/H$ and H acts freely on all orbits running through S. Finally Theorem 2 gives the desired conclusion. □

It turns out that the much more general solvable case can be treated as well, using a moderate amount of Lie theory. For the pertinent notions regarding general Lie groups and algebras we refer to [20]; the results specific to exponential Lie groups can be found in [2].

Definition 26. Let $\mathfrak{h} < \mathrm{gl}(n, \mathbb{R})$ be a Lie-subalgebra. We call $\mathfrak{h}$ *exponential* if for all $X \in \mathfrak{h}$, $\mathrm{ad}(X) : \mathfrak{h} \to \mathfrak{h}$ does not have a purely imaginary eigenvalue. □

Remark 27. It is known that all exponential Lie algebras are solvable. Given such a Lie algebra $\mathfrak{h}$, let $\widetilde{H}$ denote the associated connected, simply connected Lie group, and let $\widetilde{\exp} : \mathfrak{h} \to \widetilde{H}$ denote the exponential map. Then $\mathfrak{h}$ is exponential iff $\widetilde{\exp}$ is a diffeomorphism [2]. □

The following well-known result is a special case of [20, Theorem 3.7.3].

Lemma 28. *Let $\mathfrak{h} < \mathrm{gl}(n,\mathbb{R})$ be solvable, and (ϱ, V) an $\mathfrak{h}$-module. Then there exist $\mathbb{R}$-linear mappings $\lambda_1,\ldots,\lambda_d : \mathfrak{h} \to \mathbb{C}$ and a suitable basis $v_1,\ldots,v_d$ of V such that, in the coordinates induced by $v_1,\ldots,v_d$,*

$$\rho(X) = \begin{pmatrix} \lambda_1(X) & & & * \\ & \lambda_2(X) & & \\ & & \ddots & \\ 0 & & & \lambda_d(X) \end{pmatrix} . \tag{10}$$

The λ_i are called roots *of V.*

The following crucial definition is taken from [2].

Definition 29. Let $\mathfrak{h} < \mathrm{gl}(n,\mathbb{R})$ be exponential, and (ρ, V) an $\mathfrak{h}$-module. We call (ρ, V) a module of exponential type, if all roots of V are of the type $\lambda(X) = \psi(X)(1+i\alpha)$, with a suitable linear functional $\psi : \mathfrak{h} \to \mathbb{R}$. □

We can now formulate the central result concerning exponential solvable Lie groups.

Theorem 30. *Let $H < \mathrm{GL}(n,\mathbb{R})$ be a closed connected subgroup, with exponential Lie algebra $\mathfrak{h}$. Assume further that $\mathbb{R}^n$ is an $\mathfrak{h}$-module of exponential type (with respect to the natural action). Then :*

(a) *H is simply connected.*
(b) *For all $x \in \mathbb{R}^n$: If $\mathrm{stab}_H(x)$ is discrete, it is trivial.*
(c) *There exists a Borel fundamental domain $C \subset \widehat{\mathbb{R}^n}$ for all dual orbits.*

Proof. First note that $\mathbb{R}^n$ and $\widehat{\mathbb{R}^n}$ have the same roots, hence $\widehat{\mathbb{R}^n}$ is also of exponential type.

For (a) consider the left action of $\mathfrak{h}$ and H on $\mathrm{gl}(n,\mathbb{R})$. It is equivalent to the diagonal action on an n-fold copy of $\mathbb{R}^n$, and therefore a $\mathfrak{h}$-module of exponential type as well. Furthermore, the stabilizers $\mathrm{stab}_H(A)$ are trivial, whenever A is invertible. By [20, Theorem 3.2.7], the action lifts to an action of the simply connected covering group $\widetilde{H}$. Now [2, Theorem I.3.3] implies that all associated stabilizers in $\widetilde{H}$ are connected. For invertible A, the stabilizer coincides with the kernel of the covering map $p : \widetilde{H} \to H$. Since this kernel is also discrete, it has to be trivial. Thus H is simply connected. Now part (b) also follows from [2, Theorem I.3.3].

For the proof of part (c), [2, Theorem I.3.8] yields that every dual orbit is open in its closure. Now the desired statement follows from [6, Theorems 2.6-2.9, (5) $\Leftrightarrow$ (12)]. □

Using Theorem 30, Proposition 21 and Theorem 2 one obtains:

Corollary 31. *Let $H < \mathrm{GL}(n,\mathbb{R})$ be a closed connected subgroup, with exponential Lie algebra $\mathfrak{h}$. Then H is admissible, provided that*

(i) *There exists $h \in H$ with $|\det(h)| \neq \Delta_H(h)$;*

(ii) *there exists at least one dual orbit on which H acts locally freely;*
(iii) $\mathbb{R}^n$ *is an* $\mathfrak{h}$*-module of exponential type.*

In particular, if H is a closed, connected, abelian matrix group with real spectrum, then H is admissible if and only if (i) and (ii) hold.

6. Finitely generated abelian matrix groups with real spectrum

Consider the following situation: Suppose we are given a finite set of invertible, pairwise commuting matrices $A_1, \ldots, A_k \in \mathrm{GL}(n, \mathbb{R})$. Our task is to decide whether there exists a continuous wavelet transform associated to the matrix group $H = H_d = \langle A_1, \ldots, A_k \rangle$, and possibly to give a description of the admissible vectors. In effect, we would like to consider the wavelet system $(T_x D_{A^m} \psi)_{m \in \mathbb{Z}^k, x \in \mathbb{R}^n}$, where we use the notation $A^m = A_1^{m_1} \ldots A_k^{m_k}$, and look for conditions to reconstruct arbitrary $f \in \mathrm{L}^2(\mathbb{R}^n)$ from its scalar products from the wavelet system via the inversion formula

$$f = \sum_{m \in \mathbb{Z}^k} \int_{\mathbb{R}^n} \langle f, T_x D_{A^m} \psi \rangle \; T_x D_{A^m} \psi \; dx \; , \tag{11}$$

which is equivalent to requiring that ψ satisfy the discrete Calderón condition

$$\sum_{m \in \mathbb{Z}^k} |\widehat{\psi}(A^{-m} \omega)|^2 = 1 \; , (a.e.). \tag{12}$$

Several obstacles present themselves, in the following natural ordering:

(1) The mapping $\mathbb{Z}^k \ni m \mapsto A^m \in H$ need not be injective (it is onto by construction of H). That is, we need to check whether the generators $A_1, \ldots, A_k$ are *free* generators.
(2) H need not be discrete.
(3) Is H admissible? As we have seen above, this amounts to the existence of a group element with determinant $\neq 1$ (a property that only needs to be checked on the generators), and the existence of a measurable fundamental domain. The latter problem is quite hard to access directly.

If we assume that all A_i have positive eigenvalues, we can now decide all these questions by standard linear algebra techniques. We can compute a connected group H_c containing H_d cocompactly in a straightforward manner: First block diagonalization of the generators, then computation of their matrix logarithms, $B_i = \log A_i$, for $i = 1, \ldots, k$. Then, letting $\mathfrak{h}_c = \mathrm{span}(B_1, \ldots, B_k)$, the exponential map is a diffeomorphism onto the closed connected simply connected group $H_c \supset H_d$. Its restriction yields a group isomorphism $\langle B_1, \ldots, B_k \rangle \to H_d$, where the left-hand side is understood as additive subgroup of the vector space $\mathfrak{h}_c$. The fact that $\langle B_1, \ldots, B_k \rangle$ generates $\mathfrak{h}_c$ as a vector space implies that $\mathfrak{h}_c / \langle B_1, \ldots, B_k \rangle$ is compact, and hence H_c / H_d is compact. Moreover, the problem of deciding whether H_d is closed (i.e., discrete), is transferred to the analogous properties of $\langle B_1, \ldots, B_k \rangle$.

Now the above list of questions can be answered in the following way:

(1) The generators are free iff $B_1, \ldots, B_k \subset \mathfrak{h}_c$ are free, with respect to the additive group of that vector space. The latter means that any linear combination with integer coefficients, at least one of them nonzero, of $B_1, \ldots, B_k$ is nonzero. Clearly, this is the same as linear independence over the rationals, which can be checked by the Gauss algorithm.
(2) Assuming that $B_1, \ldots, B_k$ are free, they generate a subgroup algebraically isomorphic to $\mathbb{Z}^k$. This subgroup is closed in $\mathfrak{h}_c$ iff $\dim(\mathfrak{h}_c) = k$: Note that since the B_i span $\mathfrak{h}_c$, we always have $\dim(\mathfrak{h}_c) \leq k$, and if "$<$" holds, the countable subgroup cannot be closed by [12, 9.12]. Conversely, if the vectors $B_1, \ldots, B_k$ span $\mathfrak{h}_c$ as a vector space, their $\mathbb{Z}$-linear combinations are discrete.
By the exponential map, this statement transfers to $A_1, \ldots, A_k$. In short, H is discrete iff $(\log A_1, \ldots, \log A_k)$ are $\mathbb{R}$-linearly independent.
(3) Clearly, the determinant function is $\equiv 1$ on $\langle A_1, \ldots, A_k \rangle$ iff $|\det(A_i)| = 1$ for all $1 \leq i \leq k$.
(4) With all previous tests passed, admissibility of H is decided by checking local freeness of the action of H_c, using 21 (c).

As a result of this discussion, we find a quite striking phenomenon as we pass from a connected group to a cocompact discrete subgroup. Recall that in principle there are two obstacles to admissibility: Noncompact stabilizers and badly behaved orbit spaces. Theorem 23 points out that for connected groups with positive eigenvalues only the stabilizers condition can fail. By contrast, we know that in the discrete case this condition is trivially fulfilled, confer Lemma 3. Hence, as we pass from a connected group H_c to a discrete cocompact subgroup H_d, one obstacle turns into the other, i.e., discretization of the noncompact stabilizers in H_c results in pathological orbit spaces for H_d.

7. The general case: Partial results and counterexamples

For the general case the questions related to admissibility are not so easily answered. In this section we present some counterexamples showing that the simple sufficient criteria that apply in the real spectrum case are not valid in the general case. More precisely, we consider connected matrix groups $H = \exp(\mathfrak{h})$ and abelian matrix algebras containing H. The unit groups of the latter will allow to construct counterexamples showing that the sufficient criteria for the real spectrum case, as formulated in Corollary 31, no longer work in the general case.

The following theorem provides criteria for the existence of a measurable fundamental domain. Note the gap between the necessary and the sufficient condition.

Theorem 32. *Let $H < \mathrm{GL}(n, \mathbb{R})$ be an abelian matrix group, and let $\mathcal{A} \subset gl(n, \mathbb{R})$ be an abelian matrix algebra with $H \subset \mathcal{A}$.*

(a) *Suppose that there exists an H-invariant open subset $O \subset \mathbb{R}^n$ of full measure such that, for all $x \in O$:* $\mathrm{stab}_{\mathcal{A}^\times}(x) \cdot H$ *is closed, where $\mathcal{A}^\times$ is the unit group of $\mathcal{A}$. Then there exists a measurable fundamental domain for the H-orbits.*
(b) *Suppose that there exists an H-invariant open subset $O \subset \mathbb{R}^n$ of positive measure such that, for all $x \in O$:* $\mathrm{stab}_{\mathcal{A}^\times}(x) \cdot H$ *is not closed. Then no measurable fundamental domain of the H-orbits in $\mathbb{R}^n$ can exist.*

Proof. First note for the unit group $\mathcal{A}^\times$ that $\mathcal{A}^\times = \mathcal{A} \cap \mathrm{GL}(n, \mathbb{R})$, hence $\mathcal{A}^\times$ is an algebraic subgroup of $\mathrm{GL}(n, \mathbb{R})$. Hence, by [22, 3.1.3], the $\mathcal{A}^\times$-orbits in $\mathbb{R}^n$ are locally closed, and the natural projection $\mathcal{A}^\times \to \mathcal{A}^\times.x$ induces a homeomorphism $\mathcal{A}^\times/\mathrm{stab}_{\mathcal{A}}^\times(x) \to \mathcal{A}^\times.x$.

Now let us prove part (a): Since $H \subset \mathcal{A}^\times$, and the larger group is abelian, H acts on each orbit $\mathcal{A}^\times.x \subset \mathbb{R}^n$. Let $p_x : \mathcal{A}^\times \to \mathcal{A}^\times \cdot x$ denote the quotient map. Then $p_x^{-1}(H.x) = \mathrm{stab}_{\mathcal{A}^\times}(x) \cdot H$ is closed, and thus $H.x \subset \mathcal{A}^\times.x$ is relatively closed. Since the latter is locally closed, $H.x$ is also locally closed, for all $x \in O$. Now [6, Theorems 2.6–2.9, (5) $\Leftrightarrow$ (12)] yields the existence of a Borel set meeting each H-orbit in O precisely once.

For the proof of part (b), let G_x denote the closure of $H \cdot \mathrm{stab}_{\mathcal{A}^\times}(x)$. Then $G_x \subset \mathcal{A}^\times$ is a closed subgroup containing H. Let μ_x denote the Haar measure on $G_x/\mathrm{stab}_{G_x}(x)$, transferred to the orbit $G_x.x$ via the quotient map. Then, by definition of G_x, $H.y \subset G_x.x$ is dense for every $y \in G_x.x$, which implies that H acts ergodically on $G_x \cdot x$, if the latter is endowed with μ_x. Further, let ν_x denote the Haar measure of $\mathcal{A}^\times/\mathrm{stab}_{\mathcal{A}^\times}(x)$, transferred to $\mathcal{A}^\times.x$ by the quotient map. Then, since the $\mathcal{A}^\times$-orbits are locally closed, it follows that Lebesgue measure on O decomposes into measures equivalent to the ν_x (see [11]). On the other hand, each ν_x decomposes into the $\{\mu_y : y \in \mathcal{A}^\times.x\}$. But these are H-ergodic. Thus we have found an ergodic decomposition into measures that are not supported on single orbits. Now uniqueness of the ergodic decomposition yields that Lebesgue measure on O cannot be decomposed into measures over the H-orbits in O. Now [11, Theorem 12] and [17, Theorem 2.65] yields the desired contradiction. □

We begin with the problem of deciding whether H acts freely. Recall that, if the spectrum of the group is real, we only need to check for local a.e. freeness. The following example shows that in general this does not imply that the action is free a.e.:

Example 33. We construct a simply connected, connected closed abelian matrix group that acts locally freely but not freely almost everywhere. Let $\mathfrak{h} \subset gl(3, \mathbb{C}) \subset gl(6, \mathbb{R})$ be given as $\mathfrak{h} = \mathrm{span}(Y_1, Y_2, Y_3)$, where

$$Y_1 = \begin{pmatrix} i & 1 & \\ & i & \\ & & i \end{pmatrix}, \; Y_2 = \begin{pmatrix} 0 & & 1 \\ & 0 & \\ & & 0 \end{pmatrix}, Y_3 = \begin{pmatrix} 0 & & i \\ & 0 & \\ & & 0 \end{pmatrix}.$$

Note that only diagonal elements and nonzero off-diagonal elements are given, the remaining entries are zero. If $\mathfrak{h} = \mathfrak{h}_0 + \mathfrak{h}_1$ denotes the decomposition from Lemma 10, we see that $\mathfrak{h}_0$ is trivial, hence $H = \exp(\mathfrak{h})$ is a closed, simply connected abelian matrix group, with $\exp : \mathfrak{h} \to H$ a bijection. For $v \in \mathbb{C}^3 \equiv \mathbb{R}^6$ with $v_3 \neq 0$, one immediately

checks that the linear mapping

$$\mathbb{R}^3 \ni r \mapsto \sum_{i=1}^{3} r_i Y_i v$$

is one-to-one, and thus the action of H on the orbit of v is locally free. On the other hand, introducing the vector $s \in \mathbb{R}^3$,

$$s_1 = 1 \ , \ s_2 = -\mathrm{Re}\left(\frac{v_2}{v_3}\right) \ , \ s_3 = -\mathrm{Im}\left(\frac{v_2}{v_3}\right) \ ,$$

it follows for all $k \in \mathbb{Z}$ that $\exp\left(\sum_i 2\pi k s_i Y_i\right)(v) = v$. Thus $\mathrm{stab}_H(v)$ is not compact.
We can extend the group by including $Y_0 = \mathbf{1}_3$. The resulting larger group $\widetilde{H}$ is a closed, simply connected and connected abelian group. It fulfills the admissibility criteria (i) and (iii) from Theorem 2, acts locally freely, but nonetheless fails criterion (ii). ((iii) is fulfilled since $\mathbf{1}_3 \in \widetilde{\mathfrak{h}}$. For checking (i), use the algebra $\mathcal{A} = \mathrm{span}(Y_0, \ldots, Y_3)$ and apply Theorem 32 (a)). $\square$

Finally, an example of a simply connected closed matrix group fulfilling all admissibility criteria of Theorem 2 except (b)(i):

Example 34. Let $\mathfrak{h} \subset gl(4,\mathbb{C}) \subset gl(8,\mathbb{R})$ be defined as $\mathfrak{h} = \mathrm{span}\{Y_0, Y_1, Y_2, Y_3\}$, where $Y_0 = \mathbf{1}_4$, and

$$Y_1 = \begin{pmatrix} i\pi & & & \\ & i & 1 & \\ & & i & \\ & & & i \end{pmatrix} \ , \ Y_2 = \begin{pmatrix} 0 & & & \\ & 0 & i & \\ & & 0 & \\ & & & 0 \end{pmatrix} \ , \ Y_3 = \begin{pmatrix} 0 & & & \\ & 0 & & 1 \\ & & 0 & \\ & & & 0 \end{pmatrix} \ .$$

Then it is easily checked that $H = \exp(\mathfrak{h})$ is a closed, simply connected abelian matrix group. Again, admissibility condition (iii) is guaranteed by including Y_0 in $\mathfrak{h}$. Furthermore, it is straightforward to verify that for every $v \in \mathbb{C}^4$ such that $v_1 \neq 0 \neq v_4$ and with (v_3, iv_4) $\mathbb{R}$-linearly independent, the stabilizer of v in H is trivial.
In order to see that H does not fulfill the admissibility condition (i), we introduce the commutative matrix algebra $\mathcal{A}$ consisting of all

$$B = \begin{pmatrix} z_1 & & & \\ & z_2 & z_3 & z_4 \\ & & z_2 & \\ & & & z_2 \end{pmatrix} \in gl(4,\mathbb{C}) \ , \tag{13}$$

with $z_1, \ldots, z_4 \in \mathbb{C}$ arbitrary. Given $v \in \mathbb{C}^4$ with $v_1 \neq 0 \neq v_4$, we compute that

$$\mathrm{stab}_{\mathcal{A}^\times}(v) = \{B \text{ as in } (13) \ : \ z_1 = z_2 = 1 \ , \ z_3 v_3 + z_4 v_4 = 0\} \ .$$

We next show that, whenever iv_3, v_4 are linearly independent, then $H \cdot \mathrm{stab}_{\mathcal{A}^\times}(v)$ is not closed and then Theorem 32 (b) gives the desired conclusion. To this end, observe that $H \cdot \mathrm{stab}_{\mathcal{A}^\times}(v) = \exp(\mathfrak{g})$, with

$$\mathfrak{g} = \mathfrak{h} + \{B \text{ as in } (13) \ : \ z_1 = z_2 = 0 \ , \ z_3 v_3 + z_4 v_4 = 0\} \ . \tag{14}$$

Next, we note that for iv_3, v_4 linearly independent,

$$\mathrm{span}_{\mathbb{R}}\left(\{(i,0),(0,1)\} \cup \{(z_3,z_4) \in \mathbb{C}^2 : z_3v_3 + z_4v_4 = 0\}\right) = \mathbb{C}^2 .$$

To see this, observe that our assumptions guarantee that the $\mathbb{R}$-linear map

$$\mathbb{C}^2 \ni (z_3,z_4) \mapsto z_3v_3 + z_4v_4$$

is injective on the $\mathbb{R}$-span of $(i,0),(0,1)$. Hence this space has trivial intersection with the kernel of the linear map, and thus a simple dimension argument yields that the two spaces span all of $\mathbb{C}^2$.

But this implies that $\mathfrak{g}$ contains the matrix

$$X_0 = \begin{pmatrix} i\pi & & & \\ & i & & \\ & & i & \\ & & & i \end{pmatrix}$$

and in fact, all (complex) diagonal matrices in $\mathfrak{g}$ with purely imaginary entries are real multiples of X_0. Thus, if $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ is the decomposition from Lemma 10, then $\mathfrak{g}_0 = \mathbb{R}X_0$. But then Theorem 12 and Lemma 14 allow to conclude that $\exp(\mathfrak{g})$ is not closed. □

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Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra-Barcelona, Catalonia.

E-mail address: bruna@mat.uab.cat jcufi@mat.uab.cat marga@mat.uab.cat

Lehrstuhl A für Mathematik, RWTH Aachen, 52056 Aachen, Germany.

E-mail address: fuehr@MathA.rwth-aachen.de